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A note on a proposal for the tachyon state in vacuum string field theory

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Abstract: We discuss the proposal of Hata and Kawano for the tachyon fluctuation around a solution of vacuum string field theory representing a D25-brane. We give a conformal field theory construction of their state — a local insertion of a tachyon vertex operator on the sliver surface state, and explain why the on-shell momentum condition emerges correctly. We also show that a naive computation of the D25-brane tension using data for the three point coupling of this state gives an answer that is $(\pi^2/3)(16/27\ln 2)^3 \simeq 2.0558$ times the expected answer. We demonstrate that this problem arises because the HK state does not satisfy the equations of motion in a strong sense required for the computation of D-brane tension from the on-shell 3-tachyon coupling.

Keywords: Bosonic Strings, D-branes.

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Contents

1. Introduction

In a very stimulating paper [1] Hata and Kawano identified a particular state (hereafter refered to as the HK state) as an off-shell tachyon on the D25-brane solution in vacuum string field theory (VSFT) [2, 3, 4], and showed that the linearized field equations around the D25-brane background lead to the correct on-shell condition for the tachyon. However, when they computed the three tachyon amplitude using this on-shell state, and fixed the normalization of the action by requiring this amplitude to coincide with the known answer, the energy density $\mathcal{E}_c$ associated with the D25-brane solution did not agree with the expected answer $\mathcal{T}_{25}$ for the D25-brane tension. A refinement of their analysis has very recently appeared in [5]. In this analysis the result for the $\mathcal{E}_c/\mathcal{T}_{25}$ ratio still did not agree with the expected answer for the D25-brane tension, but came out to be about twice the expected value. This suggested the possibility that the perhaps the sliver would represent a state with two D-branes.

In both these papers the solution was constructed using the oscillator representation of the states. Although much of the calculation can be carried out analytically, the final expression for the tachyon state involves inverses and square roots of infinite dimensional matrices. The expression for the $\mathcal{E}_c/\mathcal{T}_{25}$ ratio also involves determinants of infinite dimensional matrices. Thus the final answers for these quantities have to be computed numerically by truncating the infinite dimensional matrices to finite dimensional ones.

In this note we propose a description of the HK state using the language of boundary conformal field theory (BCFT) which allows us to give an explicit simple analytic form for the tachyon state as well as a closed form for the $\mathcal{E}_c/\mathcal{T}_{25}$ ratio calculated in [5]. In particular we show that the $\mathcal{E}_c/\mathcal{T}_{25}$ ratio computed in ref. [5] is given by:

$$\frac{\mathcal{E}_c}{\mathcal{T}_{25}} = \frac{\pi^2}{3}\left(\frac{16}{27\ln 2}\right)^3 \simeq 2.0558\,. \tag{1.1}$$

This rules out the two D-brane interpretation of the sliver, and thus appears to be problematic. Nevertheless, we explain that a problem with the proposed tachyon state invalidates the computation of the $\mathcal{E}_c/\mathcal{T}_{25}$ ratio in this framework. Thus our conclusion is that the computation performed in refs. [1, 5] is not expected to produce the correct expression for the $\mathcal{E}_c/\mathcal{T}_{25}$ ratio. In a nutshell, the field equation for the on-shell tachyon state is a state whose inner product with Fock space states vanishes, but whose inner product with another tachyon state does not vanish. Thus the field equation does not hold in a strong sense for this tachyon state. The question of finding a fully consistent tachyon state thus remains open.

The paper is organized as follows. In section 2 we propose a conformal field theory description of the HK state, show that it yields the correct on-shell condition, and give numerical evidence that it corresponds to the same state discussed in refs. [1, 5]. In section 3 we analytically compute the $\mathcal{E}_c/\mathcal{T}_{25}$ ratio by naively following the procedure outlined in [1, 5] and arrive at (1.1). We then show that one of the steps in this calculation is illegal, and hence (1.1) is not a valid result.

2. Conformal field theory description of the HK state

In this section we shall give a conformal field theory description of the HK state of ref. [1], and show analytically that if we take the inner product of the linearized equations of motion around the D25-brane background with a Fock space state, we are led to the correct mass-shell condition for the tachyon on the D25-brane. Thus this gives an analytic proof of the result of ref. [1].

We shall proceed in two steps.

1. First we shall propose a conformal field theory construction of the HK state and show that it gives the correct on-shell condition when we take the inner product of the equations of motion with the Fock space state.

2. Then we present numerical evidence that our proposal for the HK state agrees with the state constructed in [1].

2.1 BCFT construction of the HK state

The VSFT action is given by [3]

$$S = -\left[\frac{1}{2}\langle\Psi|\mathcal{Q}|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi * \Psi\rangle\right], \tag{2.1}$$

where $|\Psi\rangle$ is the string field represented by a ghost number one state in the matter-ghost boundary conformal field theory (BCFT), $\langle A|B\rangle$ denotes BPZ inner product, $*$ denotes star product [8] and $\mathcal{Q}$ is an operator of ghost number one made purely of ghost fields. We have normalized $|\Psi\rangle$ and $\mathcal{Q}$ such that any overall normalization constant multiplying the action, involving the open string coupling constant, is absorbed into a redefinition of $|\Psi\rangle$

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and $\mathcal{Q}$. For definiteness we shall take the matter part of BCFT to be the one associated with a D25-brane. The equations of motion are

$$\mathcal{Q}|\Psi\rangle + |\Psi * \Psi\rangle = 0\,. \qquad (2.2)$$

As usual we look for factorized solution of the equations of motion:

$$|\Psi\rangle = |\Psi_g\rangle \otimes |\Psi_m\rangle\,, \qquad (2.3)$$

where $|\Psi_g\rangle$ and $|\Psi_m\rangle$ are the ghost and the matter parts of the solution, satisfying,

$$\mathcal{Q}|\Psi_g\rangle + |\Psi_g * \Psi_g\rangle = 0\,, \qquad (2.4)$$

and

$$|\Psi_m\rangle = |\Psi_m * \Psi_m\rangle\,. \qquad (2.5)$$

The solution $|\Psi_g\rangle$ is taken to be universal, — the same for all D-brane solutions. On the other hand, $|\Psi_m\rangle$ corresponding to different D-brane solutions differ from each other.

A solution to the matter part of the equation of motion, describing the D25-brane solution, is provided by the matter sliver, defined through the relation [6, 7, 3]

$$\langle \Xi_m|\psi\rangle = \lim_{n\to\infty} \mathcal{N}\,\langle f\circ\psi(0)\rangle_{\mathcal{C}_n} \qquad (2.6)$$

where $f(z) = \tan^{-1} z$, $|\psi\rangle$ is a state in the matter Hilbert space, $\mathcal{N}$ is a normalization factor which ensures that Ξ_m squares to itself under $*$ multiplication, and $\langle\ \rangle_{\mathcal{C}_n}$ denotes correlation function of the matter BCFT on a semi-infinite cylinder $\mathcal{C}_n$ of circumference $n\pi/2$ obtained by making the identification $\mathrm{Re}(z) \simeq \mathrm{Re}(z) + n\pi/2$ in the upper half plane. In the $n\to\infty$ limit $\mathcal{C}_n$ approaches the upper half plane. We have

$$\langle \Xi_m|\Xi_m\rangle = K V^{(26)}\,, \qquad (2.7)$$

where $V^{(26)}$ is the volume of the 26-dimensional space-time, and K is a normalization constant arising due to the conformal anomaly in the matter sector. Here we are using the normalization convention:

$$\langle k|k'\rangle = (2\pi)^{26}\delta(k+k')\,, \qquad (2.8)$$

where $|k\rangle$ denotes the Fock vacuum with momentum k. We also have the identification $(2\pi)^{26}\delta(0) = V^{(26)}$.

We now propose that the HK tachyon state is given by

$$|\Psi_g\rangle \otimes |\chi_T(k)\rangle\,, \qquad (2.9)$$

where Ψ_g, satisfying (2.4), is the same solution that appears in the construction of the classical solution describing a D-brane configuration, and

$$\langle \chi_T(k)|\,\psi\rangle \equiv \mathcal{N} \lim_{n\to\infty} n^{2k^2}\,\langle e^{ik\cdot X(n\pi/4)} f\circ\psi(0)\rangle_{\mathcal{C}_n} \qquad \forall\psi\,. \qquad (2.10)$$

In other words we simply insert a tachyon vertex operator in the middle of the (matter) sliver, at the diametrically opposite point with respect to the puncture. (Recall that in our notations the cylinder $\mathcal{C}_n$ defined as $-\pi/4 \le \mathrm{Re}(z) \le n\pi/2 - \pi/4$, $\mathrm{Im}(z) > 0$. It has circumference $n\pi/2$ and the puncture is at $f(0) = 0$. The local coordinate patch extends between the vertical lines $\mathrm{Re}(z) = \pm\pi/4$, $\mathrm{Im}(z) > 0$, which represent the two halves of the open string). The explicit factor of n^{2k^2} ensures that the state has finite overlap with Fock states except for the overall normalization factor $\mathcal{N}$ which could be infinite. The normalization factor of $\mathcal{N}$ has been included for convenience, so that the overall normalization in any formula involving $\chi_T(k)$ can be fixed by noting that for $k = 0$ we must recover the corresponding formula for Ξ_m.

Representing the state $|\chi_T(k)\rangle$ and the sliver in terms of correlation function on $\mathcal{C}_n$ has the advantage that the $*$-product / BPZ inner product of two such states has a simple geometric interpretation [7, 3]. We first cut open each cylinder $\mathcal{C}_n$ along the lines $\mathrm{Re}(z) = \pi/4$, $(2n-1)\pi/4$, representing the left and the right halves of the string respectively, to get a strip of width $(n-1)\pi/2$. We then glue the right end of the first strip to the left end of the second strip to get a bigger strip of width $(n-1)\pi$. In the case of BPZ inner product, we simply glue the left and the right ends of the resulting strip to get a cylinder $\mathcal{C}_{2n-2}$ of circumference $(n-1)\pi$. In the case of $*$-product we glue another strip of width $\pi/2$, representing the local coordinate patch, to one end of the big strip, and then glue the left and the right ends of the resulting strip to get a cylinder of width $(2n-1)\pi/2$. The final result is then expressed as a correlation function on this big cylinder. For details of this construction, see ref. [7].

In the $n \to \infty$ limit, the two operators $e^{ik\cdot X(n\pi/4)}$ and $f \circ \psi(0)$ in (2.10) are infinite distance apart, and as a result the higher the dimension of the operator in $f \circ \psi(0)$ the more supressed is its contribution to the correlator. In fact, due to the compensating factor n^{2k^2}, we will see that only the lowest dimension operator in $f \circ \psi(0)$ carrying momentum $-k$ will contribute to the correlator. More precisely, we can write

$$f \circ \psi(0) = a_\psi e^{-ik\cdot X(0)} + \text{descendents of } e^{-ik\cdot X(0)} + \text{other conformal blocks}\,, \qquad (2.11)$$

where the coefficients a_ψ depends only on ψ. The two-point function (2.10) will be non-zero only if $a_\psi \neq 0$. Indeed the primary $e^{-ik\cdot X(0)}$ is the only primary non-orthogonal to the insertion $e^{ik\cdot X}$. Thus the other conformal blocks cannot contribute. Moreover, descendents of $e^{-ik\cdot X(0)}$, which give a non-vanishing two point function for finite n, can be ignored since they have higher dimension and their contribution goes to zero for n large. Thus we have

$$\begin{aligned}\langle \chi_T(k) |\, \psi\rangle &= \lim_{n\to\infty} \Big[\mathcal{N}\, a_\psi\, n^{2k^2} \langle e^{ik\cdot X(n\pi/4)} e^{-ik\cdot X(0)}\rangle_{\mathcal{C}_n}\Big] \\ &= \lim_{n\to\infty} \left[\mathcal{N}\, a_\psi\, n^{2k^2} \left(\frac{4}{n}\right)^{2k^2} \langle e^{ikX(-1)} e^{-ikX(1)}\rangle_D\right], \end{aligned} \qquad (2.12)$$

where we have used the change of variables $w = \exp(4iz_n/n)$ to map the cylinder $\mathcal{C}_n$ parametrized by z_n to the unit disk D parametrized by w. We now record that[1]

$$\begin{aligned}\langle e^{ik\cdot X(e^{i\theta})} e^{ik'\cdot X(e^{i\theta'})}\rangle_D &= \left[2\sin\left(\frac{\theta-\theta'}{2}\right)\right]^{2k\cdot k'} (2\pi)^{26}\delta(k+k') \\ &= [d(\theta,\theta')]^{2k\cdot k'} (2\pi)^{26}\delta(k+k')\,, \end{aligned} \qquad (2.13)$$

[1]For the two point function of operators on the boundary of the unit disk we have $\langle X(z)X(w)\rangle = -\frac{1}{2}\{\ln|z-w|^2 + \ln|z - \frac{1}{\bar{w}}|^2\}$.

where $d(\theta, \theta')$ is the distance between the points $\exp(i\theta)$ and $\exp(i\theta')$. This result gives

$$\langle e^{ik\cdot X(-1)} e^{-ik\cdot X(1)} \rangle_D = \frac{1}{2^{2k^2}} V^{(26)} \,, \tag{2.14}$$

and therefore, back in (2.12) we get

$$\langle \chi_T(k) | \, \psi \rangle = 2^{2k^2} \, \mathcal{N} \, a_\psi \, V^{(26)} \,. \tag{2.15}$$

As discussed above, we can compute the star products $\Xi_m * \chi_T(k)$ and $\chi_T(k) * \Xi_m$ by method similar to the ones described in [7, 3]. In doing this we keep n large and fixed and take the $n \to \infty$ limit at the end of the computation. We get, for large n:

$$\begin{aligned}
&\langle \Xi_m * \chi_T(k) + \chi_T(k) * \Xi_m \, | \psi \rangle = \\
&\quad = \mathcal{N} \, a_\psi \, n^{2k^2} \Big[\langle e^{ik\cdot X(n\pi/4)} e^{-ikX(0)} \rangle_{\mathcal{C}_{2n-1}} + \langle e^{ik\cdot X((3n-2)\pi/4)} e^{-ikX(0)} \rangle_{\mathcal{C}_{2n-1}} \Big] \\
&\quad = \mathcal{N} \, a_\psi \, n^{2k^2} \left(\frac{4}{2n-1} \right)^{2k^2} \Big[\langle e^{ik\cdot X(i)} e^{-ikX(1)} \rangle_D + \langle e^{ik\cdot X(-i)} e^{-ikX(1)} \rangle_D \Big] \,,
\end{aligned} \tag{2.16}$$

where we have used $w = \exp(4iz_{2n-1}/(2n-1))$ to map the cylinder $\mathcal{C}_{2n-1}$ to the unit disk. Using eq. (2.13) we get, for large n,

$$\langle \Xi * \chi_T(k) + \chi_T(k) * \Xi \, | \psi \rangle = 2^{k^2+1} \, \mathcal{N} \, a_\psi \, V^{(26)} \,, \tag{2.17}$$

This is precisely the same as (2.15) if

$$k^2 = 1 \,. \tag{2.18}$$

Thus if (2.18) is satisfied, $\chi_T(k)$ satisfies the linearized equations of motion:

$$\langle \chi_T(k) | \psi \rangle = \langle (\chi_T(k) * \Xi_m + \Xi_m * \chi_T(k)) | \psi \rangle \,, \tag{2.19}$$

for any Fock space state $|\psi\rangle$. Eq. (2.18) gives the correct on-shell condition for the tachyon living on the D-brane.

Finally we note that we can replace the vertex operator $e^{ik\cdot X}$ in eq. (2.10) by any primary operator, and an analysis identical to the one carried out here will show that the corresponding state satisfies the linearized equations of motion if the primary operator has dimension one.

2.2 Comparison with the algebraic description of the HK state

Now we would like to find the oscillator form of the state $\chi_T(k)$, and compare this with the HK state constructed in ref. [1]. To this end, let us derive current conservation laws for $|\chi_T(k)\rangle$. Consider a scalar function $F(z)$ on $\mathcal{C}_n$ (more precisely on the double cover of $\mathcal{C}_n$ obtained by removing the restriction $\mathrm{Im}(z_n) \geq 0$) that is regular everywhere except for possible poles at $z = 0$. Then we claim that

$$\langle \chi_T(k) | \oint_{\mathcal{C}} d\xi \, F(f(\xi)) \, \partial X_\mu(\xi) = ik_\mu \, F_0 \, \langle \chi_T(k) | \,, \tag{2.20}$$

where $F_0 \equiv F(n\pi/4)$, $f(\xi) = \tan^{-1}\xi$, and $\mathcal{C}$ is a contour in the ξ plane surrounding the origin in the usual counterclockwise direction. This can be seen as follows. The inner product of the left hand side of eq. (2.20) with a Fock space state $|\phi\rangle$ is given by:

$$\mathcal{N} \lim_{n\to\infty} n^{2k^2} \left\langle e^{ik\cdot X(n\pi/4)} \oint_C d\xi F(f(\xi)) f\circ \partial X(\xi) f\circ\phi(0) \right\rangle_{\mathcal{C}_n} =$$
$$= \mathcal{N} \lim_{n\to\infty} n^{2k^2} \left\langle e^{ik\cdot X(n\pi/4)} \oint dz F(z) \partial X(z) f\circ\phi(0) \right\rangle_{\mathcal{C}_n}, \qquad (2.21)$$

where $z = f(\xi)$. Deforming the z contour on $\mathcal{C}_n$, since the function F has no singularity at infinity, we get just a contour surrounding the point $z = n\pi/4$ in the clockwise direction. Using the OPE

$$\partial X_\mu(z)\, e^{ik\cdot X(w)} \sim -\frac{ik_\mu}{z-w} e^{ik\cdot X(w)}, \qquad (2.22)$$

we see that we pick up a pole equal to $+iF_0 k_\mu e^{ik\cdot X(n\pi/4)}$. This gives eq. (2.20). Thus the effect of the tachyon insertion is simply to add an inhomeogenous term to the current conservation laws for the sliver surface state. Since $\mathcal{C}_n$ approaches the upper half plane as $n\to\infty$, in this limit we can take $F(z)$ to be a function on upper half plane with possible poles at the origin, and F_0 to be the value of $F(z)$ at ∞.

For $F(z) = 1$, (2.20) gives the expected conservation law for the momentum a_0:[2]

$$\langle\chi_T(k)| a_0^\mu = -\sqrt{2} k^\mu \langle\chi_T(k)|\,. \qquad (2.23)$$

Using this result, and (2.20) we can write a convenient form of the general conservation law:

$$\langle\chi_T(k)| \left(-F(\infty) a_0^\mu + \oint_0 d\xi\, F(\tan^{-1}(\xi))\, i\sqrt{2}\partial X^\mu(\xi) \right) = 0\,, \qquad (2.24)$$

where $F(z)$ is regular except for poles at $z = 0$. We can derive conservation laws for higher $a_n^\dagger$'s by taking $F(z)$ to have a pole of order n^* at $z = 0$ and converting to the local coordinate ξ using $z = \tan^{-1}\xi$. For example, taking $F(z) = 1/z^2$, we have $F(\infty) = 0$ and we obtain

$$\langle\chi_T(k)| \oint_{\mathcal{C}} d\xi \frac{1}{(\tan^{-1}\xi)^2} i\sqrt{2}\,\partial X_\mu(\xi) = 0\,, \qquad (2.25)$$

which gives

$$\left(a_2 + \frac{\sqrt{2}}{3} a_0 - \frac{1}{15} a_{-2} + \frac{32\sqrt{2}}{945} a_{-4} + \cdots \right) |\chi_T(k)\rangle = 0\,. \qquad (2.26)$$

[2] As in [1], $i\sqrt{2}\partial X(\xi) \equiv a_0\xi^{-1} + \sum_{n\neq 0}\sqrt{|n|} a_n\, \xi^{-n-1}$, with the BPZ conjugate of a_n being $(-1)^{n+1} a_{-n} = (-1)^{n+1} a_n^\dagger$. In this convention, the operator product of two ∂X in the bulk is given by $\partial X^\mu(z)\partial X^\mu(w) \simeq -\eta^{\mu\nu}/2(z-w)^2$, and that between a ∂X in the bulk and $e^{ik\cdot X}$ on the boundary is given by eq. (2.22).

By choosing $F(z) = F_n(z)$ such that

$$\sqrt{n} F_n(\tan^{-1} \xi) = \xi^{-n} + \mathcal{O}(\xi^2) \,, \tag{2.27}$$

the conservation laws (2.24) can be written in the form

$$\left(a_n + \sum_{m=1}^{\infty} S_{nm} a_m^\dagger + \hat{t}_n a_0 \right) |\chi_T(k)\rangle = 0 \,, \tag{2.28}$$

where S_{nm} is guaranteed to be the matrix that appears in the oscillator representation of the matter sliver,

$$\Xi_m = \mathcal{N} \exp \left(-\frac{1}{2} a^\dagger \cdot S \cdot a^\dagger \right) |0\rangle \,. \tag{2.29}$$

Moreover, it is clear from (2.27) that the integral term in (2.24) gives no contribution to $\hat{t}_n$ and therefore we have

$$\hat{t}_n = -(-1)^n F_n(\infty) \,. \tag{2.30}$$

Twist invariance of $\chi_T(k)$ imples that the momentum coefficients $\hat{t}_n$ vanish for odd n.

The functions F_n are readily constructed using (2.27) and power series expansions. For the first six nonvanishing cases we find

$$\begin{aligned}
\sqrt{2} F_2(z) &= \frac{1}{z^2} - \frac{2}{3} \,, \\
\sqrt{4} F_4(z) &= \frac{1}{z^4} - \frac{4}{3z^2} + \frac{26}{45} \,, \\
\sqrt{6} F_6(z) &= \frac{1}{z^6} - \frac{2}{z^4} + \frac{23}{15z^2} - \frac{502}{945} \,, \\
\sqrt{8} F_8(z) &= \frac{1}{z^8} - \frac{8}{3z^6} + \frac{44}{15z^4} - \frac{176}{105z^2} + \frac{7102}{14175} \,, \\
\sqrt{10} F_{10}(z) &= \frac{1}{z^{10}} - \frac{10}{3z^8} + \frac{43}{9z^6} - \frac{718}{189z^4} + \frac{563}{315z^2} - \frac{44834}{93555} \,, \\
\sqrt{12} F_{12}(z) &= \frac{1}{z^{12}} - \frac{4}{z^{10}} + \frac{106}{15z^8} - \frac{1360}{189z^6} + \frac{21757}{4725z^4} - \frac{6508}{3465z^2} + \frac{295272982}{638512875} \,.
\end{aligned} \tag{2.31}$$

Using (2.30) we now simply read

$$\begin{aligned}
\hat{t}_2 &= \frac{\sqrt{2}}{3} \cong 0.47140 \,, & \hat{t}_4 &= -\frac{13}{45} \cong -0.28889 \,, \\
\hat{t}_6 &= \frac{502}{945\sqrt{6}} \cong 0.21687 \,, & \hat{t}_8 &= -\frac{3551}{14175\sqrt{2}} \cong -0.17714 \,, \\
\hat{t}_{10} &= \frac{22417\sqrt{2}}{93555\sqrt{5}} \cong 0.15154 \,, & \hat{t}_{12} &= -\frac{147636491}{638512875\sqrt{3}} \cong -0.13349 \,.
\end{aligned} \tag{2.32}$$

The conservation laws (2.28) lead to the following oscillator form of $\chi_T(k)$:

$$|\chi_T(k)\rangle \sim \exp \left(-a_0 \cdot \hat{t} \cdot a^\dagger - \frac{1}{2} a^\dagger \cdot S \cdot a^\dagger \right) |k\rangle \,. \tag{2.33}$$

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L	t_2	t_4	t_6	t_8	t_{10}	t_{12}
100	0.43097	−0.25050	0.18064	−0.14271	0.11861	−0.10183
200	0.43734	−0.25656	0.18637	−0.14815	0.12381	−0.10682
300	0.44040	−0.25948	0.18914	−0.15079	0.12634	−0.10925
400	0.44234	−0.26133	0.19088	−0.15245	0.12793	−0.11079
500	0.44373	−0.26265	0.19213	−0.15364	0.12907	−0.11188
∞	0.48018	−0.29736	0.22496	−0.18487	0.15896	−0.14060

Table 1: Numerical results for t_n at different level approximation. The last row shows the interpolation of the various results to $L = \infty$, obtained via a fitting function of the form $a_0 + a_1/\ln(L)$.

The HK state of ref. [1] has exactly the same form, where the coefficients t_n, labelled without a hat to distinguish them from the predicted coefficiens $\hat{t}_n$, were computed in in terms of the matter sector Neumann coefficients of the 3-string vertex. We have evaluated the t_n's in level truncation by using $L \times L$ matrices and the numerical results are shown in table 1. We find good evidence that $\Psi_g \otimes \chi_T(k)$ coincides with the HK state.

3. Problems in the computation D25-brane tension

In this section we shall first perform a calculation similar to the ones in refs. [1, 5] to compute the D-brane tension and obtain an answer that is $\pi^2(16/27\ln 2)^3/3 = 2.056$ times the expected answer. We also show, however, that the on-shell HK state fails to satisfy the linearized equations of motion when we take the inner product of the equations of motion with the HK state itself. This invalidates the calculation of the D-brane tension.

We proceed in two steps.

1. First we show how a naive calculation of the D-brane tension yields an answer that is too big.

2. We then show that the on-shell HK state fails to satisfy the equations of motion when we take its inner product with an HK state, and as a result the calculation in the first step breaks down.

3.1 Naive computation of the D25-brane tension

In this computation we shall approximate the sliver as well as the HK state $\chi_T(k)$ defined in the previous section by defining them in terms of correlation functions on a cylinder $\mathcal{C}_n$ with finite n, and take the $n \to \infty$ limit only at the end of the computation. We define the off-shell tachyon field $T(k)$ in momentum space through the expansion:

$$|\Psi\rangle = |\Psi_g\rangle \otimes \left(|\Xi_m\rangle + \int d^{26}k\, n^{-k^2}\, T(k)\, |\chi_T(k)\rangle + \cdots \right). \tag{3.1}$$

The n^{-k^2} normalization factor has been included so that the kinetic term for near on-shell tachyons will have no n dependence. With the help of (2.4), (2.5) the action $S(|\Psi\rangle)$

computed for such a field configuration is given by:

$$\begin{aligned}
S &= S(|\Psi_g\rangle * |\Xi_m\rangle) \\
&\quad -\langle\Psi_g|\mathcal{Q}|\Psi_g\rangle\Bigg[\frac{1}{2}\int d^{26}k\, d^{26}k'\; T(k)\, T(k')\, n^{-k^2-k'^2}\times \\
&\qquad\qquad \times \langle\chi_T(k')|\Big(|\chi_T(k)\rangle - |\Xi_m * \chi_T(k)\rangle - |\chi_T(k) * \Xi_m\rangle\Big) + \\
&\qquad\qquad + \frac{1}{3}\int d^{26}k_1 d^{26}k_2 d^{26}k_3 T(k_1)T(k_2)T(k_3)n^{-k_1^2-k_2^2-k_3^2}\times \\
&\qquad\qquad\quad \times \langle\chi_T(k_1)|\chi_T(k_2) * \chi_T(k_3)\rangle\Bigg]\,.
\end{aligned} \tag{3.2}$$

Now, from eqs. (2.15) and (2.17) we have

$$|\Xi_m * \chi_T(k)\rangle + |\chi_T(k) * \Xi_m\rangle = 2^{k^2-1}|\chi_T(k)\rangle\,. \tag{3.3}$$

Substituting this into (3.2) we get:

$$\begin{aligned}
S &= S(|\Psi_g\rangle * |\Psi_m\rangle)\langle\Psi_g|\mathcal{Q}|\Psi_g\rangle\Bigg[\frac{1}{2}\int d^{26}kd^{26}k'\; T(k)T(k')n^{-k^2-k'^2}\times \\
&\qquad\qquad \times (1-2^{k^2-1})\langle\chi_T(k')|\chi_T(k)\rangle + \\
&\qquad\qquad + \frac{1}{3}\int d^{26}k_1 d^{26}k_2 d^{26}k_3\; T(k_1)T(k_2)T(k_3)n^{-k_1^2-k_2^2-k_3^2}\times \\
&\qquad\qquad\quad \times \langle\chi_T(k_1)|\chi_T(k_2) * \chi_T(k_3)\rangle\Bigg]\,.
\end{aligned} \tag{3.4}$$

We shall see in the next subsection that the use of (3.3) to obtain (3.4) is illegal — (3.3) holds in the weak sense that its inner product with any Fock space state vanishes, but it does not hold when we take its inner product with $\langle\chi_T(k')|$. Nevertheless we shall now proceed to analyze the implications of (3.4).

We shall analyze the quadratic and cubic terms separately for near on-shell tachyon $k^2 \simeq 1$. The quadratic term for near on-shell tachyon takes the form:

$$S^{(2)} \simeq \frac{1}{2}\langle\Psi_g|\mathcal{Q}|\Psi_g\rangle \ln 2 \int d^{26}k\, d^{26}k'\, n^{-k^2-k'^2}(k^2-1)T(k)T(k')\langle\chi_T(k')|\chi_T(k)\rangle\,. \tag{3.5}$$

The inner product is readily calculated

$$\begin{aligned}
\langle\chi_T(k')|\,\chi_T(k)\rangle &= K\, n^{2k^2+2k'^2}\langle e^{ik'\cdot X((n-1)\pi/2)}e^{ik\cdot X(0)}\rangle_{C_{2n-2}} \\
&= K\, n^{2k^2+2k'^2}\left(\frac{2}{n-1}\right)^{k^2+k'^2}\langle e^{ik'\cdot X(-1)}e^{ik\cdot X(1)}\rangle_D \\
&\simeq (2\pi)^{26}\, K\,\delta(k+k')\, n^{2k^2}\,.
\end{aligned} \tag{3.6}$$

The overall normalization has been fixed so that for $k = k' = 0$ we reproduce the answer for $\langle\Xi_m|\Xi_m\rangle$ given in (2.7). This gives

$$S^{(2)} \simeq \frac{1}{2}\langle\Psi_g|\mathcal{Q}|\Psi_g\rangle(2\pi)^{26}\, K\, \ln 2\int d^{26}k(k^2-1)T(k)T(-k)\,. \tag{3.7}$$

If we rescale the tachyon fluctuation field

$$\widehat{T}(k) = \Big(K \ln 2 \langle \Psi_g | \mathcal{Q} | \Psi_g \rangle\Big)^{1/2} T(k) \,, \tag{3.8}$$

then we obtain a canonical kinetic term

$$S^{(2)} \simeq \frac{1}{2}(2\pi)^{26} \int d^{26}k (k^2-1)\widehat{T}(k)\widehat{T}(-k) \,. \tag{3.9}$$

Next we turn to the cubic term in eq. (3.4). This time we compute

$$\begin{aligned}
\langle \chi_T(k_1)|\chi_T(k_2) * \chi_T(k_3)\rangle &= K n^{2(k_1^2+k_2^2+k_3^2)} \Big\langle e^{ik_1\cdot X(0)} e^{ik_2\cdot X((n-1)\pi/2)} e^{ik_3\cdot X(2(n-1)\pi/2)} \Big\rangle_{\mathcal{C}_{3n-3}} \\
&= K n^{2(k_1^2+k_2^2+k_3^2)} \left(\frac{4}{3(n-1)}\right)^{k_1^2+k_2^2+k_3^2} \times \\
&\quad \times \langle e^{ik_1\cdot X(1)} e^{ik_2\cdot X(e^{2\pi i/3})} e^{ik_3\cdot X(e^{4\pi i/3})}\rangle_D \\
&\simeq K \left(\frac{4}{3\sqrt{3}}\right)^3 n^{k_1^2+k_2^2+k_3^2} (2\pi)^{26}\delta(k_1+k_2+k_3) \,,
\end{aligned} \tag{3.10}$$

for $k_i^2 \simeq 1$ and n large. The $(\sqrt{3})^{-3}$ factor in the last step came from the correlator on the disk, computed using a simple generalization of eq. (2.13). Thus the cubic term in (3.4) now takes the following form for near on-shell momenta:

$$-\frac{K}{3}(2\pi)^{26}\langle \Psi_g|\mathcal{Q}|\Psi_g\rangle \left(\frac{4}{3\sqrt{3}}\right)^3 \int d^{26}k_1 d^{26}k_2 d^{26}k_3 \delta(k_1+k_2+k_3)\, T(k_1)T(k_2)T(k_3) \,. \tag{3.11}$$

Expressed in terms of $\widehat{T}$, this reduces to:

$$-\frac{1}{3}\Big(K\langle \Psi_g|\mathcal{Q}|\Psi_g\rangle\Big)^{-\frac{1}{2}}(2\pi)^{26}\left(\frac{4}{3\sqrt{3}\ln 2}\right)^3 \int d^{26}k_1 d^{26}k_2 d^{26}k_3 \delta(k_1+k_2+k_3)\widehat{T}(k_1)\widehat{T}(k_2)\widehat{T}(k_3) \,. \tag{3.12}$$

From this we see that the on-shell three tachyon coupling is given by:

$$g_T = \Big(K\langle \Psi_g|\mathcal{Q}|\Psi_g\rangle\Big)^{-1/2}\left(\frac{4}{3\sqrt{3}\ln 2}\right)^3 . \tag{3.13}$$

This, in turn, is related to the tension of the D25-brane via the relation [9, 10]:

$$\mathcal{T}_{25} = \frac{1}{2\pi^2 g_T^2} = \frac{1}{2\pi^2} K\langle \Psi_g|\mathcal{Q}|\Psi_g\rangle \left(\frac{3\sqrt{3}\ln 2}{4}\right)^6 . \tag{3.14}$$

On the other hand, the energy density $\mathcal{E}_c$ associated with the solution (2.3) with $|\Psi_m\rangle = |\Xi_m\rangle$ is given by

$$\mathcal{E}_c = \frac{1}{6}\langle \Psi_g|\mathcal{Q}|\Psi_g\rangle \frac{\langle \Xi_m|\Xi_m\rangle}{V^{(26)}} = \frac{K}{6}\langle \Psi_g|\mathcal{Q}|\Psi_g\rangle \,. \tag{3.15}$$

From eqs.(3.14) and (3.15) we get

$$\left(\frac{\mathcal{E}_c}{\mathcal{T}_{25}}\right)_{\text{exact}} = \frac{\pi^2}{3}\left(\frac{16}{27\ln 2}\right)^3 \simeq 2.0558 \,. \tag{3.16}$$

In [5], this ratio was determined by a direct level truncation computation for various values of the level L (table 3(a) of [5]). A large L extrapolation of their results gives[3]

$$\left(\frac{\mathcal{E}_c}{\mathcal{T}_{25}}\right)_{\rm numeric} \cong 2.0532\,, \tag{3.17}$$

in very good agreement with our exact prediction (3.16). The result is of course different from the expected answer of one. It is also different, though not very much so, from the value of two, which would suggest that the sliver describes a state with two D25-branes.

3.2 Failure of the computation

We shall now show that $\chi_T(k)$ fails to satisfy the equations of motion (3.3) when we take the inner product of this with the state $\chi_T(k')$. In other words, we want to show that

$$\langle\chi_T(k')|\chi_T(k)\rangle \neq \langle\chi_T(k')|(\chi_T(k)*\Xi_m+\Xi_m*\chi_T(k))\rangle\,. \tag{3.18}$$

We recall from (3.6) that

$$\langle\chi_T(k')|\chi_T(k)\rangle = n^{2k^2}(2\pi)^{26}K\delta(k+k')\,. \tag{3.19}$$

On the other hand,

$$\begin{aligned}
&\langle\chi_T(k')|(\Xi_m*\chi_T(k)+\chi_T(k)*\Xi_m)\rangle = \\
&\quad = K\,n^{2(k^2+k'^2)}\Big[\langle e^{ik\cdot X((n-1)\pi/2)}e^{ik'\cdot X(0)}\rangle_{\mathcal{C}_{3n-3}}+\langle e^{ik\cdot X((n-1)\pi)}e^{ik'\cdot X(0)}\rangle_{\mathcal{C}_{3n-3}}\Big]\\
&\quad = K\,n^{2(k^2+k'^2)}\left(\frac{4}{3(n-1)}\right)^{k^2+k'^2}\Big[\langle e^{ik\cdot X(e^{2\pi i/3})}e^{ik'\cdot X(1)}\rangle_D+\langle e^{ik\cdot X(e^{4\pi i/3})}e^{ik'\cdot X(1)}\rangle_D\Big]\\
&\quad = 2\,n^{2k^2}\left(\frac{4}{3}\right)^{2k^2}(\sqrt{3})^{-2k^2}\,K\,(2\pi)^{26}\delta(k+k')\,.
\end{aligned} \tag{3.20}$$

Clearly (3.19) and (3.20) are not equal for $k^2=1$. This establishes (3.18) and the failure of (3.4).

It is useful to note that the sliver itself does satisfy its projector equation of motion not only against Fock space states, but also against the sliver. In particular if we take $k=0$, in which case $\chi_T(k)$ reduces to the sliver, we see from our equations (2.15) and (2.17) that $(\Xi*\Xi+\Xi*\Xi)-2\,\Xi$ has vanishing inner product with the Fock space state. On the other hand, eqs. (3.19) and (3.20) show that $(\Xi*\Xi+\Xi*\Xi)-2\,\Xi$ also has vanishing inner product with the sliver.[4]

[3] We extrapolated the quantity H to infinite level with a fit of the form $a_0+a_1/\log(L)$, then computed (3.17) using eq. (5.19) of [1].

[4] Since for the matter part alone the normalization of the sliver involves infinite factors, in order to make a precise statement one needs to work with the sliver of the combined matter ghost CFT. In this case the sliver Ξ, normalized so that $\langle\Xi|Y(i)|\Xi\rangle=1$, with $Y(i)=\frac{1}{2}c\partial c\partial^2c(i)$, satisfies the condition that $\langle\phi|Y(i)|(\Xi-\Xi*\Xi)\rangle=0$ when $|\phi\rangle$ is a Fock space state of ghost number 0, and also when $|\phi\rangle$ is the sliver $|\Xi\rangle$. On the other hand, if we define the state $|\chi_T(k)\rangle$ as in eq. (2.10) with $\langle\cdot\rangle_{\mathcal{C}_n}$ now denoting the correlation function in the combined matter ghost CFT, then we shall find that $\langle\phi|Y(i)(|\Xi_m*\chi_T(k)\rangle+|\chi_T(k)*\Xi_m\rangle-2^{k^2-1}|\chi_T(k)\rangle)$ vanishes for $k^2=1$ as long as $|\phi\rangle$ is a Fock space state, but that it does not vanish if $|\phi\rangle$ is of the form $|\chi_T(k')\rangle$.

4. Discussion

Given the failure of the HK tachyon state to satisfy a strong form of the equation of motion, as indicated in eq. (3.18), we see that the computation of the $\mathcal{E}_c/\mathcal{T}_{25}$ ratio outlined in refs. [1, 5] is not valid. This leaves open the question as to whether the identification of the HK state with the tachyon fluctuation around the D25-brane background could still be valid. The tachyon certainly does satisfy the linearized equation of motion around the D25-brane solution for $k^2 = 1$ in a weak sense, i.e. when we consider the inner product of the equations of motion with a Fock space state. On the other hand it fails to satisfy the equations of motion when we consider its inner product with the tachyon state itself. Clearly a more satisfactory construction of the tachyon fluctuation will be one for which the inner product of the equations of motion with any state yields the correct on-shell condition for the tachyon.

We note that the problem discussed here is a special case of a more general ambiguity that arises in dealing with correlation functions of states of the type (2.9). In particular, when computing various correlation functions we have regularized the sliver by replacing it by a finite n wedge state $|n\rangle$, and *have chosen the same value of* n *for the sliver* $|\chi_T(k)\rangle$ *and* $|\chi_T(k')\rangle$. If we had chosen different values of n for regularizing different states, we would have gotten different answers for these correlators. Thus in general we need to be very careful in dealing with such states.

One could of course compute these correlation functions also using oscillator methods as in ref. [1]. One might wonder if the oscillator computation suffers from similar ambiguities. To this end we note that the expressions involving these correlation functions involve inverse powers of the matrix $(1 + 3M^{11})$, where $M^{rs} = CV^{rs}$, with $C_{mn} = (-1)^m \delta_{mn}$ the charge conjugation matrix, and V^{rs}_{mn} the matter Neumann coefficients. Due to the presence of a $(-1/3)$ eigenvalue of the matrix M^{11} [5, 11, 12], the matrix $(1 + 3M^{11})$ is actually singular, and as a result various formal identities break down in the presence of inverse powers of $(1 + 3M^{11})$. In particular, although formally $[M^{rs}, M^{r's'}]$ vanishes, it turns out that $[M^{rs}, M^{r's'}](1 + 3M^{11})^{\alpha}$ does not vanish in general for $\alpha < 0$, and definitely does not vanish for $\alpha \leq -3/2$. Thus different expressions which are formally identical have different values in practice. This, in turn, makes the correlation functions involving states of type (2.9) ambiguous. The eigenvalue spectrum of the matrix M^{11} will be discussed in a separate paper [12].

An alternative proposal for computing the ratio $\mathcal{E}_c/\mathcal{T}_{25}$ was given in [5] in which it was noted that the expression for the ratio $\mathcal{E}_c/\mathcal{T}_{25}$ in the oscillator approach contains a term which is formally zero, but because of the presence of inverse powers of the matrix $(1 + 3M^{11})$, it does not vanish. In analysing this term, the authors expand the expression for $\mathcal{E}_c/\mathcal{T}_{25}$ in a power series expansion in $(1 + 3M_{11})$ and for those terms containing powers larger than $-3/2$, simplify the expressions by using the formal identities satisfied by the matrices M^{rs}. This gives a simpler expression for $\mathcal{E}_c/\mathcal{T}_{25}$. The numerical result for this modified expression also comes out to be close to two. The precise correspondence between this modified expression and the conformal field theory description is not entirely clear to us at present.

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